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Asymptotic behavior of measures of dependence for ARMA(1,2) models with stable innovations. Stationary and non-stationary coefficients

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Asymptotic behavior of measures of dependence for ARMA(1,2) models with stable innovations. Stationary and non-stationary coefficients.

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Abstract

We derive the asymptotic behavior of two measures of dependence (Codifference and Covariation) for ARMA(1,2) models with symmetric α-stable innovations and non-stationary coefficients.

1 Introduction

Definition 1 *The system ARMA(1,q) is given by the following formula:*

$$X_n - b_n X_{n-1} = \sum_{j=0}^{q-1} a_{n-j}\xi_{n-j}, \tag{1}$$

where the innovations (ξ_n) are independent, symmetric α-stable with the scale parameter 1,i.e. with the characteristic function given by:

$$Eexp(i\theta\xi_n) = exp(-|\theta|^{\alpha}), 0 < \alpha \leq 2. \tag{2}$$

Moreover the coefficients (a_n) and (b_n) are nonzero and complex for all $n \in Z$.

In [7] Weron and Wylomanska show the conditions, which give a bounded solutions of system ARMA(1,q), where the innovations are uncorrelated complex random variables with mean 0 and variance 1. In this case we have three conditions, which give bounded solution of the system, but only two give its unique solution. Therefore we consider two cases:

I. $\sup_q |B_0^q| = \infty$

II. $\sup_q |B_q^0|^{-1} = \infty$.

In this paper $X_n = Y$ (X_n-the sequence of random variables, Y- the random variable) means $\lim_n ||X_n - Y|| = 0$ and $B_r^s = \prod_{j=r}^{s} b_j$ (with the convention that $B_r^s = 1$ if r>s). In the analysis we consider the system ARMA(1,2) given by the following equation:

$$X_n - b_n X_{n-1} = a_n\xi_n + a_{n-1}\xi_{n-1}, \tag{3}$$

where the innovations and coefficients have the same properties like in the general model.

Definition 2 *Measures of dependence of jointly symmetric $\alpha-$stable random variables X_1 and X_2 [6].*

- **Covariation $\mathbf{CV}(X_1, X_2)$** *of X_1 on X_2 defined for $1 < \alpha \leq 2$ is the real number*

$$CV(X_1, X_2) = \int_{S_2} s_1 s_2^{<\alpha-1>} \Gamma(ds), \tag{4}$$

where Γ is the spectral measure of the random vector (X_1, X_2), $z^{<p>} = |z|^{p-1}\bar{z}$,

- **Codifference $\mathbf{CD}(X_1, X_2)$** *of X_1 on X_2 defined for $0 < \alpha \leq 2$ equals*

$$CD(X_1, X_2) = lnEexp\{i(X_1 - X_2)\} - lnEexp\{iX_1\} - lnEexp\{-iX_2\}. \tag{5}$$

Unlike the codifference, the covariation is not symmetric, but is linear in the first argument. The covariation is closely related to the quantity $E[X_1 X_2^{<p-1>}]$. In this paper the norm $||X||$, where X is a symmetric $\alpha-$stable random variable, is defined $||X|| = (CV(X, X))^{1/\alpha}$ (the covariation norm [6]). If $\alpha = 2$ the following identities hold

$$CV(X_1, X_2) = \frac{1}{2} Cov(X_1, X_2),$$

$$CD(X_1, X_2) = Cov(X_1, X_2).$$

In contrast to [3, 4] we study here time series ARMA(1,2) with non-stationary coefficients. The presented new general proofs use the form of the bounded solution of ARMA(1,2) model from [7]. This least us to consider two separately cases described by Condition 1 and Condition 2. The main results are included in Theorem 1 and Theorem 2. They give the asymptotic behavior of the quotations $CD(X_n, X_{n+k_q})/CV(X_n, X_{n+k_q})$ and $CD(X_n, X_{n-k_q})/CV(X_n, X_{n-k_q})$ for both cases, respectively. Formulas 10 and 20 generalize the earlier result of Nowicka in [4]. However formulas 11 and 21 produce a new type of result even in the case of stationary coefficients.

2 Condition 1

If $\sup_q |B_0^q| = \infty$, then there exist sequence (k_q) of positive integers such that $\lim_q |B_1^{k_q}| = \infty$, and for all $n \in Z$ $\lim_q |B_{n+1}^{n+k_q}| = \infty$. In this case the solution of (3) is given by (see [7]):

$$X_n = -\lim_q \left[\sum_{j=1}^{k_q-1} \frac{a_{n+j}\xi_{n+j}}{B_{n+1}^{n+j}}\left(1 + \frac{1}{b_{n+j+1}}\right) + \frac{a_n \xi_n}{b_{n+1}} + \frac{a_{n+k_q}\xi_{n+k_q}}{B_{n+1}^{n+k_q}}\right].$$

If we denote:

$$c_n(j) = \begin{cases} 0 & j > k_q \\ -\frac{a_{n+k_q}}{B_{n+1}^{n+k_q}} & j = k_q \\ -\frac{a_{n+j}}{B_{n+1}^{n+j}}\left(1 + \frac{1}{b_{n+j+1}}\right) & 0 < j < k_q \\ -\frac{a_n}{b_{n+1}} & j = 0 \\ 0 & j < 0, \end{cases}$$

then we can write:

$$X_n = \lim_q [\sum_{j=n}^{k_q+n} c_n(j-n)\xi_j].$$

In this case the covariation of X_n on X_{n+k_q} is given by:

$$CV(X_n, X_{n+k_q}) = c_n(k_q)c_{n+k_q}^{<\alpha-1>}(0) = \frac{|a_{n+k_q}|^\alpha}{|b_{n+k_q+1}|^{\alpha-2}\overline{b_{n+k_q+1}}B_{n+1}^{n+k_q}}. \quad (6)$$

However the covariation of X_{n+k_q} on X_n has the following form:

$$CV(X_{n+k_q}, X_n) = c_{n+k_q}(0)c_n^{<\alpha-1>}(k_q) = \frac{|a_{n+k_q}|^\alpha}{b_{n+k_q+1}|B_{n+1}^{n+k_q}|^{\alpha-2}\overline{B_{n+1}^{n+k_q}}}. \quad (7)$$

If the coefficients satisfy condition 1, then the codifference of X_n on X_{n+k_q} is given by the formula:

$$CD(X_n, X_{n+k_q}) = |c_n(k_q)|^\alpha + |c_{n+k_q}(0)|^\alpha - |c_n(k_q) - c_{n+k_q}(0)|^\alpha.$$

Therefore:

$$CD(X_n, X_{n+k_q}) = |\frac{a_{n+k_q}}{B_{n+1}^{n+k_q}}|^\alpha + |\frac{a_{n+k_q}}{b_{n+k_q+1}}|^\alpha - |\frac{a_{n+k_q}}{B_{n+1}^{n+k_q}} - \frac{a_{n+k_q}}{b_{n+k_q+1}}|^\alpha.$$

The codifference takes the form:

$$CD(X_n, X_{n+k_q}) = |\frac{a_{n+k_q}}{b_{n+k_q+1}}|^\alpha(1 + |\frac{b_{n+k_q+1}}{B_{n+1}^{n+k_q}}|^\alpha - |1 - \frac{b_{n+k_q+1}}{B_{n+1}^{n+k_q}}|^\alpha). \quad (8)$$

The codifference is symmetric, therefore:

$$CD(X_n, X_{n+k_q}) = CD(X_{n+k_q}, X_n),$$

If $1 < \alpha \leq 2$ and $\sup_q |B_{n+1}^{n+k_q}| = \infty$, then the following fact is true for all $n \in Z$:

$$\lim_{k_q\to\infty} \frac{B_{n+1}^{n+k_q}}{b_{n+k_q+1}}[1 + |\frac{b_{n+k_q+1}}{B_{n+1}^{n+k_q}}|^\alpha - |1 - \frac{b_{n+k_q+1}}{B_{n+1}^{n+k_q}}|^\alpha] = \alpha. \quad (9)$$

Theorem 1 *Suppose (X_n) is the solution of system (3) and $\sup_q |B_1^q| = \infty$, then for $1 < \alpha \leq 2$ and for all $n \in Z$ the following are fulfilled:*

$$\lim_{k_q\to\infty} \frac{CD(X_n, X_{n+k_q})}{CV(X_n, X_{n+k_q})} = \alpha \quad (10)$$

$$\lim_{k_q\to\infty} \frac{|b_{n+k_q+1}|^{\alpha-2}CD(X_{n+k_q}, X_n)}{|B_{n+1}^{n+k_q}|^{\alpha-2}CV(X_{n+k_q}, X_n)} = \alpha \quad (11)$$

if $CV(X_n, X_{n+k_q}) \neq 0$ and $CV(X_{n+k_q}, X_n) \neq 0$.

PROOF: We use formulas (6), (8) and (9) to compute (10) . We obtain (11) from (7), (8) and (9).

□

For $\alpha = 2$ naturally we have:

$$\lim_{k_q\to\infty} \frac{CD(X_n, X_{n+k_q})}{CV(X_n, X_{n+k_q})} = \frac{CD(X_{n+k_q}, X_n)}{CV(X_{n+k_q}, X_n)} = \alpha.$$

Corollary 1 *If we take $n - k_q$ instead n in formulas (10) and (11), then we obtain the following formulas:*

$$\lim_{k_q\to\infty} \frac{CD(X_{n-k_q}, X_n)}{CV(X_{n-k_q}, X_n)} = \alpha \tag{12}$$

and

$$\lim_{k_q\to\infty} \frac{|b_{n+1}|^{\alpha-2} CD(X_n, X_{n-k_q})}{|B^n_{n-k_q+1}|^{\alpha-2} CV(X_n, X_{n-k_q})} = \alpha. \tag{13}$$

Remark 1 *System ARMA(1,2) with the time varying coefficients given by the formula:*

$$X_n - b_n X_{n-1} = a_0(n)\xi_n + a_1(n)\xi_{n-1} \tag{14}$$

do not have property (10), because for the system we obtain the following formulas:

$$CV(X_n, X_{n+k_q}) = \frac{a_0(n+k_q)}{B^{n+k_q}_{n+1}} \left| \frac{a_1(n+k_q+1)}{b_{n+k_q+1}} \right|^{\alpha-2} \overline{\frac{a_1(n+k_q+1)}{b_{n+k_q+1}}}$$

and

$$CD(X_n, X_{n+k_q}) = \left|\frac{a_0(n+k_q)}{B^{n+k_q}_{n+1}}\right|^{\alpha} + \left|\frac{a_1(n+k_q+1)}{b_{n+k_q+1}}\right|^{\alpha} - \left|\frac{a_0(n+k_q)}{B^{n+k_q}_{n+1}} - \frac{a_1(n+k_q+1)}{b_{n+k_q+1}}\right|^{\alpha}.$$

Therefore the asymptotic behaviour of the measures CV and CD has the following forms:

$$\lim_{k_q\to\infty} \frac{|a_1(n+k_q+1)|^{\alpha-2}\overline{a_1(n+k_q+1)} CD(X_n, X_{n+k_q})}{|a_0(n+k_q)|^{\alpha-2}\overline{a_0(n+k_q)} CV(X_n, X_{n+k_q})} = \alpha.$$

Example 1 *We consider now ARMA(1,2) model given by the equation:*

$$X_n + 2X_{n-1} = \sqrt{2}^n \xi_n + \sqrt{2}^{n-1} \xi_{n-1}.$$

The coefficients (b_n) satisfy Condition 1, i.e. $\sup_q |B^q_0| = sup_q 2^q = \infty$. On Figure 1 we show the plot of $\frac{CD(X_n, X_{n+k_q})}{\alpha CV(X_n, X_{n+k_q})}$ for $k_q = 0, 1, ..50$, $n = 10$ and $\alpha = 1.2$ and $\alpha = 1.5$.

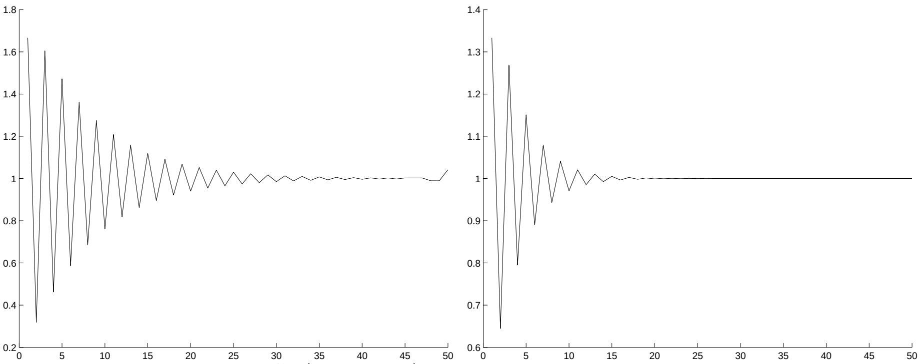

Figure 1. The plot of $\frac{CD(X_n,X_{n+k_q})}{\alpha CV(X_n,X_{n+k_q})}$ *for* $\alpha = 1.2$ *(left) and* $\alpha = 1.5$ *(right).*

Remark 2 *For the comparison we consider system ARMA(1,2) with the stationary coefficients, given by the equation:*

$$X_n - b_1 X_{n-1} = a_1 \xi_n + a_2 \xi_{n-1}, \tag{15}$$

where the innovations and coefficients are like in the definition 1. In this case the formulas (10) and (11) have the following forms respectivelly:

$$\lim_{k_q \to \infty} \frac{CD(X_{-k_q}, X_0)}{CV(X_{-k_q}, X_0)} = \alpha$$

and

$$\lim_{k_q \to \infty} \frac{CD(X_0, X_{-k_q})}{|b_1|^{(\alpha-2)(k_q-1)} CV(X_0, X_{-k_q})} = \alpha.$$

3 Condition 2

If $\sup_q |B_q^0|^{-1} = \infty$, then there exist a sequence (k_q) of positive integers such that $\lim_q |B_{-k_q}^0|^{-1} = \infty$, and for all $n \in Z$ $\lim_q |B_{n-k_q}^n|^{-1} = \infty$. In this case the solution of system (3) is given by the formula (see [7]):

$$X_n = \lim_q [\sum_{j=1}^{k_q-1} a_{n-j} B_{n+2-j}^n (1 + b_{n-j+1}) \xi_{n-j} + a_{n-k_q} B_{n-k_q+2}^n \xi_{n-k_q} + a_n \xi_n].$$

If we assume:

$$c_n(j) = \begin{cases} 0 & j > k_q \\ a_{n-k_q} B_{n-k_q+2}^n & j = k_q \\ a_{n-j} B_{n+2-j}^n (1 + b_{n+1-j}) & 0 < j < k_q \\ a_n & j = 0 \\ 0 & j < 0, \end{cases}$$

then the solution of (3) takes the form:

$$X_n = \lim_q [\sum_{j=n-k_q}^{n} c_n(n-j) \xi_j].$$

If the coefficients b_n fulfill condition 2, then the covariation of X_n on X_{n-k_q} has the form:

$$CV(X_n, X_{n-k_q}) = c_n(k_q) c_{n-k_q}^{<\alpha-1>}(0) = |a_{n-k_q}|^\alpha B_{n-k_q+2}^n. \tag{16}$$

Furthermore, the covariation of X_{n-k_q} on X_n is given by the formula:

$$CV(X_{n-k_q}, X_n) = |a_{n-k_q}|^\alpha |B_{n-k_q+2}^n|^{\alpha-2} \overline{B_{n-k_q+2}^n}. \tag{17}$$

And the codifference of X_n on X_{n-k_q} is given by the following:

$$\begin{aligned} CD(X_n, X_{n-k_q}) &= |c_n(k_q)|^\alpha + |c_{n-k_q}(0)|^\alpha - |c_n(k_q) - c_{n-k_q}(0)|^\alpha = \\ &= |a_{n-k_q} B_{n-k_q+2}^n|^\alpha + |a_{n-k_q}|^\alpha - |a_{n-k_q} B_{n-k_q+2}^n - a_{n-k_q}|^\alpha. \end{aligned} \tag{18}$$

If $1 < \alpha \leq 2$ and $\sup_q |B_{n-k_q}^n|^{-1} = \infty$, then the following equation is fulfilled

$$\lim_{k_q \to \infty} \frac{1}{B_{n-k_q+2}^n} (1 + |B_{n-k_q+2}^n|^\alpha - |1 - B_{n-k_q+2}^n|^\alpha) = \alpha. \tag{19}$$

Theorem 2 *Suppose (X_n) is the solution of system (3) and $\sup_q |B_q^0|^{-1} = \infty$, then for $1 < \alpha \leq 2$ and for all $n \in Z$ the following is fulfilled:*

$$\lim_{k_q \to \infty} \frac{CD(X_n, X_{n-k_q})}{CV(X_n, X_{n-k_q})} = \alpha \tag{20}$$

$$\lim_{k_q \to \infty} \frac{|B_{n-k_q+2}^n|^{\alpha-2} CD(X_{n-k_q}, X_n)}{CV(X_{n-k_q}, X_n)} = \alpha \tag{21}$$

if $CV(X_n, X_{n-k_q}) \neq 0$ and $CV(X_{n-k_q}, X_n) \neq 0$.

PROOF: We use formulas (16), (18) and (19) to compute (20) . We obtain (21) from (17), (18) and (19).

□

Corollary 2 *We take in formulas (20) and (21) $n + k_q$ instead n and obtain:*

$$\lim_{k_q \to \infty} \frac{CD(X_{n+k_q}, X_n)}{CV(X_{n+k_q}, X_n)} = \alpha \tag{22}$$

$$\lim_{k_q \to \infty} \frac{|B_{n+2}^{n+k_q}|^{\alpha-2} CD(X_n, X_{n+k_q})}{CV(X_n, X_{n+k_q})} = \alpha. \tag{23}$$

Example 2 *We consider system ARMA(1,2) model given by the equation:*

$$X_n + \frac{1}{2} X_{n-1} = \sqrt{2}^n \xi_n + \sqrt{2}^{n-1} \xi_{n-1}.$$

The coefficients b_n satisfy Condition 1, i.e. $\sup_q |B_q^0|^{-1} = \sup_q 2^q = \infty$. *On Figure 2 we show the plot of* $\frac{CD(X_n, X_{n-k_q})}{\alpha CV(X_n, X_{n-k_q})}$ *for* $k_q = 0, 1, ..50$, $n = 10$ *and* $\alpha = 1.2$ *and* $\alpha = 1.5$.

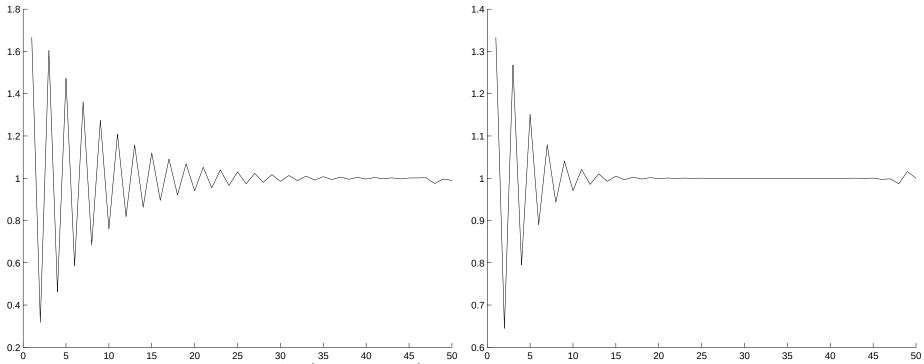

Figure 2. The plot of $\frac{CD(X_n, X_{n-k_q})}{\alpha CV(X_n, X_{n-k_q})}$ *for* $\alpha = 1.2$ *(left) and* $\alpha = 1.5$ *(right).*

Remark 3 *If X_n is the solution of system (15), then formulas (20) and (21) have the following forms:*

$$\lim_{k_q \to \infty} \frac{CD(X_{k_q}, X_0)}{CV(X_{k_q}, X_0)} = \alpha$$

and

$$\lim_{k_q \to \infty} \frac{CD(X_0, X_{k_q})}{|b_1|^{(\alpha-2)(k_q-1)} CV(X_0, X_{k_q})} = \alpha.$$

In [4] there are given the formulas to the covariation and codifference of stationary ARMA(p,q) models and the autor described asymptotic behavior of the measures of dependence. The main result of [4] is given in Corollary 1. It is shown, that the relation $\lim_{n\to\infty} \frac{CD(X_n, X_0)}{CV(X_n, X_0)} = \alpha$ holds, when some conditions are fulfilled. Therefore, if we assume that the coefficients in ARMA(1,2) model depend on the time (non-stationary ARMA model), then we obtain similar result like in the stationary case.

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